$\hat{X}|x\rangle = x|x\rangle$. eigenstate

eigenvalue (real number. $(-\infty,+\infty)$).

$\langle x'|x\rangle = \delta(x,x') = \begin{cases} \infty, & x=x' \\ 0. & x\neq x' \end{cases}$

积分.

$\int dx\ \delta(x-x') = 1.$

x是空间的一组基 $\int dx\,|x\rangle\langle x| = \mathbb{I}$.

将φ状态用x表示.

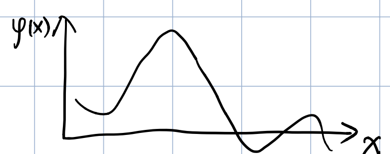
$\langle x|\varphi\rangle = \varphi(x)$. a wave function.

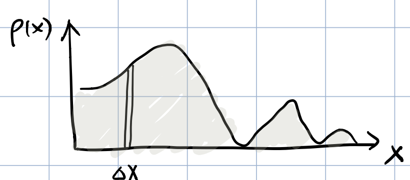
$|\varphi\rangle = \mathbb{I}|\varphi\rangle = \int |x\rangle\langle x|\varphi\rangle dx = \int \varphi(x)\,|x\rangle dx.$

$\langle\varphi|\varphi\rangle = \int \langle\varphi|x\rangle\langle x|\varphi\rangle dx = \int \varphi^*(x)\varphi(x)\,dx = \int |\varphi(x)|^2 dx = 1$

一个状态需用函数表示概率分布时，它需要正交归一.

$\rho(x) = |\varphi(x)|^2$. $\rho(x)$ is the density probality of measuring a particle is the position of x.





$\int \rho(x)\,dx = 1.$

$\int$ 概率 × 位置 = 平均值.

$\int x\rho(x)\,dx = \boxed{\langle x\rangle = \int \varphi^*(x)\,x\,\varphi(x)\,dx} = \int \langle\varphi|x\rangle\, x\, \langle x|\varphi\rangle dx$

$\langle x\rangle$的定义是 $\int \varphi^*(x)\,x\,\varphi(x)\,dx.$

$\because \int |x\rangle x\langle x|\,dx = \hat{X}$. operator.

$\therefore$ 上式 $= \langle\varphi|\hat{X}|\varphi\rangle$.

$\hat{X} = \sum$ (本征值 × 投影算符).

$\int x\rho(x)\,dx$

$= \int x\varphi^*(x)\varphi(x)\,dx$. (三个都是函数)

$= \int x\langle\varphi|x\rangle\langle x|\varphi\rangle dx.$

Translation $(\hat{T})$ 平移算符.

定义：$T(\Delta x)|x\rangle = |x+\Delta x\rangle$.

classic: $p = m\dot{x} = m\cdot\frac{dx}{dt}$.

$\therefore |x+\Delta x\rangle - |x\rangle = \Delta x\, k|x\rangle$. $\Rightarrow$ $|x+\Delta x\rangle \neq |x\rangle + |\Delta x\rangle$. 原因：

$\hat{X}|x+\Delta x\rangle = (x+\Delta x)|x+\Delta x\rangle$.

$\hat{X}(|x\rangle + |\Delta x\rangle) = x|x\rangle + \Delta x|\Delta x\rangle$

$T(\Delta x)$作用在$|x\rangle$上，使$|x\rangle$移动了Δx.

因为 $\mathbb{I}|x\rangle = |x\rangle$.

$\therefore$ 当 Δx 足够小时 $T(\Delta x) = \mathbb{I} - k\Delta x + O(\Delta x)$. $\xrightarrow{\text{更改}}$ $\mathbb{I} - ik\Delta x + O(\Delta x)$.

① $T(\Delta x_1)\overbrace{T(\Delta x_2)|x\rangle} = T(\Delta x_1)|x+\Delta x_2\rangle$

$= |x+\Delta x_1+\Delta x_2\rangle = T(\Delta x_1+\Delta x_2)|x\rangle.$

$T(\Delta x_1)T(\Delta x_2) = T(\Delta x_1+\Delta x_2).$

② $T^{-1}(\Delta x) = T(-\Delta x)$

$T^{-1}(\Delta x)\,T(\Delta x) = \mathbb{I}.$

$T(-\Delta x)T(\Delta x) = \mathbb{I}$

$= T(0) = \mathbb{I}$

$T(0)|x\rangle = |x\rangle.$

③ $T^{+}(\Delta x)$

$T(\Delta x)|x\rangle = |x+\Delta x\rangle.$

$\Updownarrow$ DC

$\langle x|T^{+}(\Delta x) = \langle x+\Delta x|$

在矢空间存在时,
左矢空间也存在相应的变换.

$\underbrace{\langle x+\Delta x|x+\Delta x\rangle}_{=\mathbb{I}} = \langle x|T^{+}(\Delta x)T(\Delta x)|x\rangle. = \underbrace{\langle x|x\rangle}_{=\mathbb{I}}\,|T^{+}\cdot T|.$

$T^{+}(\Delta x)\,T(\Delta x) = \mathbb{I}.$

相反的移动.

$\mathbb{I} = \underline{T(-\Delta x)T(\Delta x)}|x\rangle = |x\rangle.$

定义为$T(\Delta x)$的逆操作 $T^{-1}(\Delta x)$.

$\because T(\Delta x)|x\rangle = |x+\Delta x\rangle.$

$\therefore \langle x|T^{+}(\Delta x) = \langle x+\Delta x|$

内积:

$\langle x|\underline{T^{+}(\Delta x)T(\Delta x)}|x\rangle = \underbrace{\langle x+\Delta x|x+\Delta x\rangle}_{=1}$

$\therefore \boxed{T^{+}(\Delta x)T(\Delta x) = \mathbb{I}}$ 幺正性.

$\because T^{-1}(\Delta x)T(\Delta x) = \mathbb{I}$

$\therefore T^{+}(\Delta x) = T^{-1}(\Delta x)$

$T(\Delta x)T^{+}(\Delta x) = \mathbb{I}.$

$\therefore (\mathbb{I}-k\Delta x)(\mathbb{I}-k^{+}\Delta x) = \mathbb{I}.$

$\mathbb{I} - k\Delta x - k^{+}\Delta x + \cancel{kk^{+}\Delta x} = \mathbb{I}$

此时不满足. 因此假设时需引入i.

$(\mathbb{I}-ik\Delta x)(\mathbb{I}+ik^{+}\Delta x) = \mathbb{I}.$

$\mathbb{I} - ik\Delta x + ik^{+}\Delta x = \mathbb{I}.$

$\Rightarrow i(k^{+}-k)\Delta x = 0.$

$\Rightarrow$ K是一个厄密矩阵.

$k^{+} = k.$

$T(\Delta x) = \mathbb{I} - ik\Delta x + O(\Delta x^2) = 1 - \frac{iP}{\hbar}\Delta x.$

泰勒展开.

$\hat{k}$ is the generator of the translation Operator.

$T(0) = \mathbb{I}$

$T(\Delta x_1)T(\Delta x_2) = (\mathbb{I}-ik\Delta x_1)(\mathbb{I}-ik\Delta x_2)$

$= \mathbb{I} - ik\Delta x_1 - ik\Delta x_2 + \cancel{(ik)^2\Delta x_1\Delta x_2}$

$= \mathbb{I} - ik(\Delta x_1+\Delta x_2)$

$= T(\Delta x_1+\Delta x_2).$

$$T^{\dagger}(\Delta x) = (\mathbb{I} - ik\Delta x)^{\dagger} = \mathbb{I} + ik^{\dagger}\Delta x.$$

$k^{\dagger} = k.$ Hermitian.

$$T^{\dagger}(\Delta x)\, T(\Delta x) = (1 + ik\Delta x)(1 - ik\Delta x) = \mathbb{I}.$$

继续求k的表示：

$$\langle x|\hat{T}(\Delta x)|\psi\rangle = \int \langle x|\hat{T}(\Delta x)|x'\rangle\langle x'|\psi\rangle\, dx'$$

$$= \int \langle x|x'+\Delta x\rangle\langle x'|\psi\rangle\, dx'$$

$\langle x|x'\rangle = \delta(x-x').$

$$= \int \delta(x - x' - \Delta x)\,\psi(x')\,dx'$$

$$= \psi(x - \Delta x)$$

$$= \psi(x) - \Delta x \frac{\partial}{\partial x}\psi(x).$$

斜率

连续谱中 δ 函数的定义：

$$\Rightarrow \int \psi(x)\,\delta(x-x')\,dx = \psi(x').$$

$$\langle x|\hat{T}(\Delta x)\psi\rangle = \langle x|\mathbb{I} - ik\Delta x|\psi\rangle$$

$$= \psi(x) - i\Delta x\,\langle x|k|\psi\rangle$$

$$\cancel{\psi(x)} - \cancel{\Delta x}\frac{\partial}{\partial x}\psi(x) = \cancel{\psi(x)} - i\cancel{\Delta x}\langle x|k|\psi\rangle.$$

$$\frac{\partial}{\partial x}\psi(x) = i\ \langle x|k|\psi\rangle.$$

$$\parallel$$

$$\langle x|\frac{\partial}{\partial x}|\psi\rangle.$$

$$\langle x|\frac{\partial}{\partial x}|\psi\rangle = \langle x|ik|\psi\rangle$$

$$\frac{\partial}{\partial x} = ik.$$

$$k = -i\frac{\partial}{\partial x}.$$

$\langle x|k = -i\langle x|\frac{\partial}{\partial x}$

$\hat{p} = \hbar\hat{k} = -i\hbar\frac{\partial}{\partial x}.$

$\langle x|\hat{p} = -i\hbar\langle x|\frac{\partial}{\partial x}.$ 代回上式.

目的：求x和p的对易关系：

$[x,p] = i\hbar.$

$\langle x|[\hat{x},\hat{p}]|\varphi\rangle = i\hbar\langle x|\varphi\rangle$

$\boxed{\langle x|\hat{x} = x.}$

证明：

$= \langle x|\hat{x}\hat{p}-\hat{p}\hat{x}|\varphi\rangle$

$= \langle x|\hat{x}\hat{p}|\varphi\rangle - \langle x|\hat{p}\hat{x}|\varphi\rangle$

$\langle x|\hat{p} = -i\hbar\frac{\partial}{\partial x}\langle x|.$

$= x\langle x|\hat{p}|\varphi\rangle - \langle x|\hat{p}\hat{x}|\varphi\rangle$

$\therefore = -i\hbar x\langle x|\frac{\partial}{\partial x}|\varphi\rangle + \underbrace{i\hbar\langle x|\frac{\partial}{\partial x}\hat{x}|\varphi\rangle}$

$= \cancel{-i\hbar x\frac{\partial}{\partial x}\varphi(x)} + i\hbar\frac{\partial}{\partial x}(x\varphi(x)) \Rightarrow \cancel{xi\hbar\frac{\partial}{\partial x}\varphi(x)} + i\hbar\varphi(x)$

$= i\hbar\varphi(x)$

$\langle x|[x,p] = i\hbar\langle x|$

$[x,p] = i\hbar.$

$\hat{p}|p\rangle = p|p\rangle.$

$\int dp\, |p\rangle\langle p| = \mathbb{I}.$

$$\int dx\,|x\rangle\langle x| = \mathbb{I}.$$

$$\int dx\,dp\,|x\rangle\langle x|\,|p\rangle\langle p| = \mathbb{I}.$$

$$\int dx\,dp\,dp'\,|p'\rangle\left[\langle p'|\,|x\rangle\langle x|\,|p\rangle\right]\langle p| = \mathbb{I}.$$

求 $\langle x|p\rangle$　　p表示中的状态,要换到x中的表示.　　$\langle x|$.

$$\langle x|\hat{p}|p\rangle = \langle x|p|p\rangle$$

$$\therefore -i\hbar\frac{\partial}{\partial x}\langle x|p\rangle = p\,\langle x|p\rangle.$$

$$-i\hbar\frac{\partial}{\partial x}f(x) = p\,f(x)$$

$$\langle x|p\rangle = f(x) = c\,e^{\frac{ip}{\hbar}x}$$

对x做偏分: $f'(x) = \frac{\partial}{\partial x}e^{\frac{ip}{\hbar}x} = \frac{ip}{\hbar}e^{\frac{ip}{\hbar}x}.$

$\varphi(p)$

$$\langle p|\varphi\rangle = \int\langle p|x\rangle\langle x|\varphi\rangle dx = \int dx\,\varphi(x)\,c^* e^{-\frac{ip}{\hbar}x}$$

$$\langle x|x'\rangle = \delta(x-x') = \int dp\,\langle x|p\rangle\langle p|x'\rangle = \int c\,e^{\frac{ip}{\hbar}x}\cdot c^* e^{-\frac{ip}{\hbar}x'}\,dp$$

$$= |c|^2\int dp\,e^{\frac{ip(x-x')}{\hbar}}$$

$$= |c|^2\cdot 2\pi\hbar\,\delta(x-x').$$

$$C = \frac{1}{\sqrt{2\pi\hbar}}$$

$$\varphi(p) = \int \varphi(x) e^{\frac{-ipx}{\hbar}} dx \Big/ \sqrt{2\pi\hbar}$$

$$\varphi(x) = \langle x|\varphi\rangle = \int \langle x|p\rangle\langle p|\varphi\rangle dp$$

傅立叶变换 $= \int \varphi(p) e^{\frac{ip}{\hbar}x} dp \Big/ \sqrt{2\pi\hbar}$.

$\varphi(x) = \langle \varphi|x\rangle$.

$\varphi(p) = \langle p|\varphi\rangle$.

$$= \int \langle p|x\rangle\langle x|\varphi\rangle dx$$

$$= \int \langle x|p\rangle^* \varphi(x) dx$$

$$= \int \varphi(x)\cdot \frac{1}{\sqrt{2\pi\hbar}} e^{-\frac{ip}{\hbar}x} dx$$

$$= \varphi(p)$$

Gaussian wave Packet

目的：取高斯函数验证不确定关系.

$$\Delta x \cdot \Delta p \geq \frac{\hbar}{2}$$

$$\langle x|\varphi\rangle = \varphi(x) = \frac{1}{\pi^{\frac{1}{4}}\sigma^{\frac{1}{2}}} \exp\left(ikx - \frac{x^2}{2\sigma^2}\right)$$

$$\int |\varphi(x)|^2 dx = \int \frac{1}{\pi^{\frac{1}{2}}\sigma} \exp\left(-\frac{x^2}{\sigma^2}\right) dx = 1.$$

高斯分布：$\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = 1.$

μ 是期望. σ^2 是方差.

计算出 μ 和 σ. 可对相似积分直接求解.

python 画图.

$$\sum_i P(x_i)\, dx = 1.$$

$\langle x\rangle$ 是 x 在波函数上的平均值.
$\langle x^2\rangle$ 是 x^2 在波函数上的平均值.

取高斯函数 $\varphi(x)$.

$$\langle x\rangle = \langle \varphi_{(x)}|x|\varphi_{(x)}\rangle = \int dx \langle\varphi|x\rangle x \langle x|\varphi\rangle$$

$$= \int x|\varphi(x)|^2 dx = 0.$$

$$\langle x^2\rangle = \langle\varphi_{(x)}|x^2|\varphi_{(x)}\rangle = \underbrace{\int dx\cdot x^2 |\varphi(x)|^2}_{\int \varphi^*(x)\, x^2\, \varphi(x)\, dx} = \frac{\sigma^2}{2}.$$ (宽度)

$$\Delta x^2 = \langle x^2\rangle - \langle x\rangle^2.$$

$$\varphi(p) = \frac{1}{\sqrt{2\pi\hbar}} \int dx\, \varphi(x)\, e^{\frac{-ip}{\hbar}x}.$$

$$= \frac{1}{\sqrt{2\pi\hbar}} \int dx \frac{1}{\pi^{\frac{1}{4}}\sigma^{\frac{1}{2}}} \exp\left(ikx - \frac{x^2}{2\sigma^2}\right) \exp\left(-\frac{ip}{\hbar}x\right)$$

$$\Downarrow$$

$$\exp\left(-\frac{x^2}{2\sigma^2} + \left(ik - \frac{ip}{\hbar}\right)x\right)$$

$$\exp\left[-\frac{1}{2\sigma^2}\left(x - \left(ik - \frac{ip}{\hbar}\right)\sigma^2\right)^2 + \frac{1}{2\sigma^2}\left(\left(ik - \frac{ip}{\hbar}\right)\sigma^2\right)^2\right.$$

$$= C\exp\left(\frac{1}{2\sigma^2}\left(\left(ik - \frac{ip}{\hbar}\right)\sigma^2\right)^2\right)$$

$$= C\exp\left(-\frac{\sigma^2}{2}\left(k - \frac{p}{\hbar}\right)^2\right)$$

$\langle p\rangle = \hbar k.$

$\langle p^2\rangle - \langle p\rangle^2 = \frac{\hbar^2}{2\sigma^2}.$

$$C = \frac{1}{\sqrt{2\pi\hbar}} \cdot \frac{1}{\pi^{\frac{1}{4}}\sigma^{\frac{1}{2}}} \times \sqrt{2\sigma^2\pi}.$$

python 画图.

$$\varphi(p) = \frac{1}{\sqrt{2\pi\hbar}} \int \varphi(x) e^{\frac{-ip}{\hbar}x}\, dx$$

$$= \frac{1}{\sqrt{2\pi\hbar}} \sum_{i=0}^{N} \varphi(x_i)\, e^{\frac{-ip}{\hbar}x_i}$$

$$= \frac{1}{\sqrt{2\pi\hbar}} \sum_{i=0}^{N} \varphi(x_i)\, e^{\frac{-i\omega\cdot 2\pi n}{N}x_i}$$

$[-X, X]$.

$N dx = 2X.$

$$\Delta p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} = \frac{\hbar}{\sqrt{2}\,\sigma}$$

$$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \frac{\sigma}{\sqrt{2}}$$

$$\Delta x \Delta p = \frac{\hbar}{2} \geq \frac{\hbar}{2}.$$ uncertainty principle.

$[A, B] = i\hbar$. 对任意 A 和 B.

$$\Delta A \Delta B \geq \frac{\hbar}{2}$$