Gaussian Wave Packet 高斯波包.

已知: $\langle x|\varphi\rangle = \frac{1}{\pi^{1/4}\sigma^{1/2}} \exp\left(ikx - \frac{x^2}{2\sigma^2}\right)$.

$$|\varphi(x)|^2 = \frac{1}{\pi^{1/2}\sigma} \exp\left(-\frac{x^2}{\sigma^2}\right).$$

$$\int_{-\infty}^{+\infty} |\varphi(x)|^2 dx = 1.$$

$$\langle p|\varphi\rangle = \int \langle p|x\rangle\langle x|\varphi\rangle dx \qquad \left(\int |x\rangle\langle x| dx = 1\right).$$

即 $\varphi(p) = \int \frac{1}{\sqrt{2\pi\hbar}} \exp\left(\frac{ipx}{\hbar}\right)\varphi(x)\, dx = C\cdot\exp\left[\left(k-\frac{p}{\hbar}\right)^2\cdot\frac{\sigma^2}{2}\right]$

$$\langle p\rangle = \int \varphi^*(p)\cdot p\, \varphi(p)\, dp = \hbar k.$$

$$\langle p^2\rangle - \langle p\rangle^2 = \frac{\hbar^2}{2\sigma^2}.$$

$$\langle x\rangle = 0.$$

$$\langle x^2\rangle - \langle x\rangle^2 = \frac{\sigma^2}{2}.$$

$$\Delta p \Delta x = \sqrt{\langle p^2\rangle - \langle p\rangle^2} - \sqrt{\langle x^2\rangle - \langle x\rangle^2} =$$

$\therefore \Delta p \Delta x \geqslant \frac{\hbar}{2}$. uncertainty principle.

Uncertainty and its relationship of operators that don't commute $[P, Q] = PQ - QP \neq 0$, arbierary ket $|\varphi\rangle$.

$$|\psi\rangle = (Q + i\lambda P)|\varphi\rangle \xleftrightarrow{DC} \langle\psi| = \langle\varphi| Q - i\lambda P.$$

$$\langle\psi|\psi\rangle \geqslant 0$$

$$= \langle\varphi|(Q - i\lambda P)(Q + i\lambda P)|\varphi\rangle$$

$$= \underbrace{\langle\varphi|Q^2|\varphi\rangle}_{C} - \underbrace{i\lambda\langle\varphi|PQ\rangle\varphi + i\lambda\langle\varphi|QP|\varphi\rangle}_{-i\lambda\underbrace{\langle\varphi|[P,Q]|\varphi\rangle}_{B}} + \lambda^2\underbrace{\langle\varphi|P^2|\varphi\rangle}_{A}.$$

$$\langle\psi|\psi\rangle = C - i\lambda B + \lambda^2 A \geqslant 0.$$

$$= A\left(\lambda^2 - i\frac{B}{A}\lambda - \frac{B}{4A^2}\right) + A\cdot\frac{B}{4A^2} + C$$

$$= A\left(\lambda - \frac{iB}{2A}\right)^2 + \frac{B^2}{4A} + C \geq 0.$$

取一λ，使 $\lambda - \frac{iB}{2A} = 0$.

$$\therefore \frac{B^2}{4A} + C \geq 0$$

即 $AC \geq -\frac{B^2}{4}$.

$$\langle \varphi | P^2 | \varphi \rangle \langle \varphi | Q^2 | \varphi \rangle \geq -\frac{1}{4}\left(\langle \varphi | [P, Q] | \varphi \rangle\right)^2.$$

$P' = P - \langle \varphi | P | \varphi \rangle$. $Q' = Q - \langle \varphi | Q | \varphi \rangle$. $[P', Q'] = [P, Q]$.

$$\underbrace{\langle \varphi | P'^2 | \varphi \rangle}_{\Delta P^2} \langle \varphi | Q'^2 | \varphi \rangle \geq -\frac{1}{4}\left(\langle \varphi | [P', Q'] | \varphi \rangle\right)^2.$$

$$\Delta P^2 \quad \Delta Q^2 \geq -\frac{1}{4}\left(\langle \varphi | [P, Q] | \varphi \rangle\right)^2.$$

(P的方差)

$$\Delta P \Delta Q \geq \frac{1}{2} \left| \langle \varphi | [P, Q] | \varphi \rangle \right|.$$

$$[x, p] = i\hbar.$$

$$\Delta x \Delta p \geq \frac{1}{2} i\hbar.$$

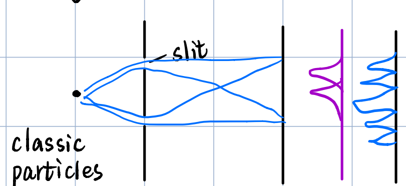
Heisenberg relationship.

$$\Delta P^2 = \langle \varphi | P^2 | \varphi \rangle - \langle \varphi | P | \varphi \rangle^2$$

$$\langle \varphi | P'^2 | \varphi \rangle = \langle \varphi | P^2 - 2P\langle P \rangle + \langle P \rangle^2 \varphi \rangle = \langle \varphi | P^2 | \varphi \rangle - \langle P \rangle^2.$$

Copenhagen interpretation 哥本哈根解释.

Yang's donble-slit experiment

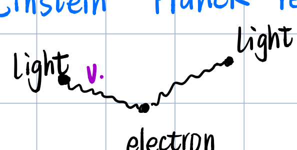


electron (quantum particle) can be wave-like.

出现的高度做为出现的概率.

phase concellation.

Einstein - Planck relation 光电效应



$E = \hbar\omega = h\nu$. 当频率达到ν以上时，才能激发出电子.

$p = \hbar k$. k is wave vector.

quantum light can behave particle like. 能量分立.

#1 Wave-particle duality.

$|\varphi\rangle$ at fixed time t_0, state of system is defined by specifying a ket $|\varphi(t_0)\rangle$ belonging to a state space $\mathcal{E}$ (Hilbert space).

$$\langle\varphi|\varphi\rangle = 1.$$

#1 inner product

#2 Linear algebra

#3 normalizable

#2 Every measurable physical quanting A, is describled by an operator, acting in $\mathcal{E}$. This operator is an "Observable".

#3 The only possible results of the measurment of a physical quanting A is the eigenvalues of the corresponding obserable $\hat{A}$.

x. p.

$\sigma_x . \sigma_y . \sigma_z . \pm 1 . \frac{\hbar}{2}$

#4 When physical quantity A is measured on a system in the normalized state $|\varphi(t_0)\rangle$. the probability of obtaining non-degerate eigenvalue.

归一化的状态. 非简并的.

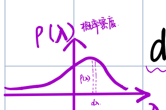
$$|\langle U_n|\varphi\rangle|^2 \qquad \hat{A}|U_n\rangle = \lambda_n|U_n\rangle.$$

A的本征态. Un不简并.

$$\hat{A} = \sum_n \lambda_n |U_n\rangle\langle U_n|.$$

$$P_n = \Sigma |\langle U_n|\varphi\rangle|^2.$$ 计算λ_n的概率.

连续: $\hat{A} = \int d\lambda \cdot \lambda\, |U_\lambda\rangle\langle U_\lambda|$ A算符的矩阵表示. 分立: $\hat{A} = \Sigma \lambda_n |U_n\rangle\langle U_n|$.



$$\underbrace{dP(\lambda)}_{\text{概率}} = \underbrace{|\langle U_\lambda|\varphi\rangle|^2}_{\text{纵坐标高}} \underbrace{d\lambda}_{\text{横坐标宽}}.$$

#5 The Time Evolution of the State is governed by the Schrodinger Equation. $i\hbar \frac{\partial}{\partial t}|\varphi\rangle = \underset{\sim}{H}|\varphi\rangle$. 状态随时间的演化由薛定谔方程控制.

Hamiltonian

位置 $\dot{x} = \frac{dx}{dt}$. $\dot{x} = \frac{\partial H}{\partial p}$. $H = T + V$. (动能 + 势能).

动量 $\dot{p} = \frac{dp}{dt}$. $\dot{p} = -\frac{\partial H}{\partial x}$. 正则方程.

经典力学中用于计算粒子的运动方程.

Schrodinger Equation.

$$i\hbar \frac{\partial}{\partial t} |\varphi\rangle = \hat{H} |\varphi\rangle$$

作用在 $\langle x|$ 上.

$$i\hbar \frac{\partial}{\partial t} |\varphi(x) = \langle x|\hat{H}|\varphi\rangle$$

Free particle 自由粒子没有势能.

$p = -i\hbar \frac{d}{dx}$. $p^2 = -\hbar^2 \frac{d^2}{dx^2}$.

$$H = T = \frac{p^2}{2m}.$$

$\langle x| p = -i\hbar \frac{\partial}{\partial x} \langle x|$.

$H = \frac{p^2}{2m} = \frac{-\hbar^2}{2m} \frac{d^2}{dx^2}$.

$\langle x| p^2 = -\hbar^2 \frac{\partial^2}{\partial x^2} \langle x|$.

$$i\hbar \frac{\partial}{\partial t} \varphi(x) = -\frac{\hbar^2}{2m} \cdot \frac{\partial^2}{\partial x^2} \varphi(x)$$

$$H = T + V$$

$$\langle x| H = \langle x| (T+V) = \left(-\frac{\hbar^2}{2m} \cdot \frac{\partial^2}{\partial x^2} + V(x)\right) \langle x|$$

$V(x)$ is the pontential energy operator in real space representation.

$$i\hbar \frac{\partial}{\partial t} \varphi(x,t) = -\frac{\hbar^2}{2m} \cdot \frac{\partial^2}{\partial x^2} \varphi(x,t)$$

只对时间求导

Seperation of varoble. 分离变量.

只对空间求导.

$\boxed{\varphi(x,t) = f(x) g(t)}$ ansatz.

$$f(x)\, i\hbar \frac{\partial}{\partial t} g(t) = -\frac{\hbar^2}{2m} g(t) \frac{\partial^2}{\partial x^2} f(x)$$

只与t有关 $\underbrace{\frac{i\hbar \frac{\partial}{\partial t} g(t)}{g(t)}} = \underbrace{\frac{-\frac{\hbar^2}{2m} \cdot \frac{\partial^2}{\partial x^2} f(x)}{f(x)}}_{\text{只与x有关}} = \lambda$. λ is a number or const.

$$i\hbar \frac{\partial}{\partial t} g(t) = \lambda g(t)$$

$$-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} f(x) = \lambda f(x)$$

$$\begin{cases} g(t) = C e^{-\frac{i\lambda}{\hbar} t}. \\ f(x) = C e^{ikx}. \end{cases}$$

$$-\frac{\hbar^2}{2m} C_{(k)} (-k^2) e^{ikx} = \lambda C e^{ikx}$$

function

$$\lambda = \frac{\hbar^2 k^2}{2m} \quad \underline{\underline{\text{definition}}} \quad \hbar w.$$

$$\psi(x,t) = \overset{g(t)\cdot f(x)}{C_{(k)}} e^{ikx - \frac{i\lambda}{\hbar} t}$$ plane wave 平面波.

$$= C_{(k)} e^{i(kx - wt)} . \qquad \underline{w = \frac{\lambda}{\hbar} = \frac{\hbar k^2}{2m}}$$

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}} \int C(k) e^{\frac{i(kx-wt)}{\hbar}} dk. \qquad w = \frac{\hbar k^2}{2m}.$$ 傅立叶变换.

wave Packet

$C(k)$ is the weight of the plane wave.

$$C(k) = \begin{cases} C & k = k_0. \\ 0 & \text{otherwise}. \end{cases}$$

$$\psi(k,t) = C(k_0) \exp i(k_0 x - w_0 t). \qquad w_0 = \frac{\hbar k_0^2}{2m}.$$

$$k_0 x - w_0 t = 0.$$

画图: phase velocity $= v_p = \frac{x}{t} = \frac{w_0}{k_0} = \frac{\hbar k_0}{2m}$.

$$\psi(x,t) = C \exp i(k_0 x - w_0 t)$$
$$+ C \exp[i(k_0 + \Delta k) x - (w_0 + \Delta w) t]$$
$$+ C \exp[i(k_0 - \Delta k) x - (w_0 - \Delta w) t]$$

欧拉公式 $e^{i\theta} = \cos\theta + i\sin\theta$.

$$= C \exp(i(k_0 x - w_0 t)) [1 + \exp(i\Delta k x - i\Delta w t) + \exp(-i\Delta k x + i\Delta w t)]$$
$$\Downarrow$$
$$(1 + 2\cos(\Delta k x - \Delta w t))$$

$=0.$

群速度

$V_g = \text{group velocity} = \frac{x}{t} = \frac{\partial \omega}{\partial k}.$

群速度：整体形状的运动

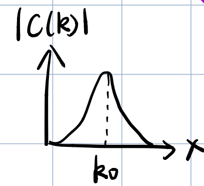
（最高点的速度）

相速度：关注一个点的运动.

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}} \int C(k) \cdot e^{i(kx-\omega t)} \, dk$$

$$C(k) = |C(k)| e^{i\varphi(k)}$$

$|C(k)|$ is only large around $k=k_0$



代回.

泰勒展开.（忽略高阶项）

$$\theta(k) = \theta(k_0) + \frac{\partial \theta}{\partial k}\Big|_{k=k_0} (k-k_0)$$

$$\omega(k) = \omega(k_0) + \frac{\partial \omega}{\partial k}\Big|_{k=k_0} (k-k_0)$$

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}} \int dk \, |C(k)| e^{i(\theta_0 + \frac{\partial\theta}{\partial k}|_{k=k_0} (k-k_0))} e^{ikx} e^{-i\omega k_0 t} \times e^{-i \frac{\partial \omega}{\partial k}|_{k=k_0} (k-k_0) t}$$

下标有0都是常数.

$$= \underbrace{\frac{1}{\sqrt{2\pi}} e^{i(\theta_0 - \frac{\partial\theta}{\partial k}|_{k=k_0} \cdot k_0)} e^{i(\frac{\partial\omega}{\partial k}|_{k=k_0} k_0 t)}}_{\text{常数}} \int dk \, \underbrace{|C(k)|}_{\text{波矢为k的波的权重}} e^{i\frac{\partial\theta}{\partial k} k} \times e^{ikx} \times e^{-i\frac{\partial\omega}{\partial k} \cdot kt}$$

e指数为0

振幅在 $\frac{\partial\theta}{\partial k}\Big|_{k=k_0} \cdot k + kx - \frac{\partial\omega}{\partial k}\Big|_{k=k_0} kt = 0$ 时最大.

平衡量.

$$k\left(\frac{\partial\theta}{\partial k}\Big|_{k=k_0} + x - \frac{\partial\omega}{\partial k}\Big|_{k=k_0} t\right) = 0$$

$$V_{group} = \frac{x}{t} = \frac{\partial\omega}{\partial k}\Big|_{k=k_0} \qquad \exp(ik(\varphi + x - V_{group} t)) = 0.$$

一组波峰值的传播速度.

$$= \frac{\hbar}{m} k \quad \longleftarrow \quad \omega = \frac{\hbar k^2}{2m}.$$

$$V_p = \frac{\omega}{k} = \frac{\hbar k}{2m}.$$

单个波的传播速度（相位改变速度）.

相速度.

classic Mechnies

$$\boxed{\begin{aligned} p &= mv \\ E &= \frac{p^2}{2m} \end{aligned}}$$

$$p = mv = mV_{group} = m \cdot \frac{\hbar}{m} \cdot k = \hbar k.$$

$$E = \frac{p^2}{2m} = \frac{(\hbar k)^2}{2m} = \frac{\hbar^2 k^2}{2m}.$$

$$C(k) = \delta(k_0 - k)$$

$$\psi(x,t) = \exp(ik_0 x - wt)$$

$$\psi(x, t=0) = \delta(x - x_0)$$

$$= \frac{1}{\sqrt{2\pi}} \int \sqrt{2\pi}\, e^{-ikx_0} e^{ikx} dk.$$

$$C(k) = \sqrt{2\pi} \exp(-ikx_0)$$