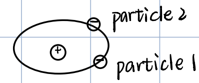
Addition of Augular Momenta

#1 $\vec{L} = \vec{R} \times \vec{P}$



$\vec{L} = \sum_{i}^{N} \vec{L_i}$

$V(r)$ central potential

$\vec{L} = L(\theta, \varphi, \cancel{r})$

$[\vec{L}, H] = 0$

$H_1 = \frac{P_1^2}{2m} + V(r_1)$

$H_2 = \frac{P_2^2}{2m} + V(r_2)$

$[\vec{L_1}, H_1] = 0$

$[\vec{L_2}, H_2] = 0$

$H = H_1 + H_2 + V(\vec{r_1} - \vec{r_2})$

$[\vec{L_1}, H] \neq 0 \qquad [\vec{L}, H] = 0$

$\vec{L} = \vec{L_1} + \vec{L_2}$

#2

spin-orbit Coupling

$H = \frac{P^2}{2m} + \mu \vec{L} \cdot \vec{S} + V(r)$

$\vec{J} = \vec{L} + \vec{S} \Rightarrow$ spin AM

$\vec{L} \downarrow$ orbital AM

Consider

$S_1 \quad S_2$

$\vec{S} = \vec{S_1} + \vec{S_2}$

$[S_{1i}, S_{2j}] = 0$

$i, j, k = x, y, z$

$[S_{1i}, S_{1j}] = i\hbar \varepsilon_{ijk} S_{1k}$

$[S_{2i}, S_{2j}] = i\hbar \varepsilon_{ijk} S_{2k}$

$$S_j = S_{1j} + S_{2j} \quad \begin{cases} S_x = S_{1x} + S_{2x} \\ S_y = S_{1y} + S_{2y} \\ S_z = S_{1z} + S_{2z} \end{cases}$$

$\Downarrow$

$[S_i, S_j] = i\hbar \varepsilon_{ijk} S_k.$

$[S^2, S_z] = 0.$

CSCO

$[S_z, S_{1z}] = 0$

$[S_z, S_{2z}] = 0$

S^2

S_z

S_{1z}

S_{2z}

$$\begin{cases} S^2 |s, m\rangle = s(s+1)\hbar^2 |s, m\rangle \\ S_z |s, m\rangle = m\hbar |s, m\rangle \end{cases}$$

$S_z = S_{1z} + S_{2z}$

$$\Rightarrow \begin{cases} S_{1z} |s, m\rangle = m_1\hbar |s, m\rangle \\ S_{2z} |s, m\rangle = m_2\hbar |s, m\rangle \end{cases}$$

$m_1\hbar + m_2\hbar = m\hbar.$

$|s, m, m_1, m_2\rangle$

$m_1 + m_2 = m.$

$\underset{S_{1z}.}{\Uparrow} \quad \underset{S_{2z}.}{\Uparrow}$

$|m_1, m_2\rangle$

$|S_{1z}, S_{2z}\rangle = |\uparrow\downarrow\rangle, |\uparrow\uparrow\rangle, |\downarrow\downarrow\rangle, |\downarrow\uparrow\rangle$

$|\frac{1}{2}, -\frac{1}{2}\rangle$

$= |j_1, m_1; j_2, m_2\rangle = |j_1, m_1\rangle \otimes |j_2, m_2\rangle$

$S_z |m_1, m_2\rangle = (m_1 + m_2)\hbar |m_1, m_2\rangle$

$= m\hbar |m_1, m_2\rangle$

$m = -1, 0, 1.$

$$S_z = \begin{pmatrix} \overset{\frac{1}{2}+\frac{1}{2}}{1} & & & \\ & 0 & & \\ & & 0 & \\ & & & \underset{}{\overset{-\frac{1}{2}-\frac{1}{2}}{-1}} \end{pmatrix} \begin{pmatrix} |\uparrow\uparrow\rangle \\ |\uparrow\downarrow\rangle \\ |\downarrow\uparrow\rangle \\ |\downarrow\downarrow\rangle \end{pmatrix}$$

$J_z |m_1, m_2\rangle$

$= (J_{1z} \otimes \mathbb{I}_2 + \mathbb{I}_1 \otimes J_{2z})(|j_1, m_1\rangle \otimes |j_2, m_2\rangle)$

$= J_{1z} |j_1, m_1\rangle \otimes \mathbb{I}_2 |j_2, m_2\rangle + \mathbb{I}_1 |j_1, m_1\rangle \otimes J_{2z} |j_2, m_2\rangle$

$= m_1\hbar |j_1, m_1\rangle \otimes |j_2, m_2\rangle$

$+ |j_1, m_1\rangle \otimes m_2\hbar |j_2, m_2\rangle$

$S^2|m_1, m_2\rangle =$ $= (m_1\hbar + m_2\hbar)\, \underline{|j_1, m_1\rangle \otimes |j_2, m_2\rangle}$

$= (m_1+m_2)\hbar\, |m_1, m_2\rangle$

解：$S^2 = (\vec{S}_1+\vec{S}_2)^2 = S_1^2 + S_2^2 + 2\vec{S}_1\cdot\vec{S}_2$

$\vec{S}_1\vec{S}_2 = S_{1x}S_{2x} + S_{1y}S_{2y} + S_{1z}S_{2z}$

$$\begin{cases} S_{1+} = S_{1x} + iS_{1y} \\ S_{1-} = S_{1x} - iS_{1y} \\ S_{2+} = S_{2x} + iS_{2y} \\ S_{2-} = S_{2x} - iS_{2y} \end{cases}$$

$\therefore S_{1x}S_{2x} + S_{1y}S_{2y} = S_{1+}S_{2-} + S_{1-}S_{2+}$

$S^2|\uparrow\uparrow\rangle = (S_1^2 + S_2^2 + \underline{S_{1+}S_{2-}} + \underline{S_{1-}S_{2+}} + 2S_{1z}S_{2z})|\uparrow\uparrow\rangle$

$= (\frac{3}{4}\hbar^2 + \frac{3}{4}\hbar^2 + 0 + 0 + 2\times\frac{1}{2}\times\frac{1}{2}\hbar^2)|\uparrow\uparrow\rangle$

$(\frac{1}{2}(\frac{1}{2}+1)\hbar^2)$

$= 2\hbar^2|\uparrow\uparrow\rangle$

$S^2|\uparrow\downarrow\rangle = (S_1^2 + S_2^2 + S_{1+}S_{2-} + \boxed{S_{1-}S_{2+}} + 2S_{1z}S_{2z})|\uparrow\downarrow\rangle$

$\frac{3}{4} + \frac{1}{4} = 1$

$= (\frac{3}{4}\hbar^2 + \frac{3}{4}\hbar^2 + 0)|\uparrow\downarrow\rangle + \hbar^2|\downarrow\uparrow\rangle + 2\times(\frac{1}{2})(-\frac{1}{2})\hbar^2|\uparrow\downarrow\rangle$

$= \underline{\hbar^2(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)}$

$S_- = \sqrt{j(j+1) - m(m-1)}\,\hbar = \sqrt{\frac{1}{2}\cdot\frac{3}{2} - \frac{1}{2}(-\frac{1}{2})}$

$S_+ = \sqrt{j(j+1) - m(m+1)}\,\hbar = \sqrt{\frac{1}{2}\times\frac{3}{2} - (-\frac{1}{2})\frac{1}{2}} = 1.$

$S^2|\downarrow\uparrow\rangle = \hbar^2(|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle)$

$S^2|\downarrow\downarrow\rangle = 2\hbar^2|\downarrow\downarrow\rangle$

$$S^2 = \begin{pmatrix} 2 & & & \\ & \boxed{\begin{matrix} 1 & 1 \\ 1 & 1 \end{matrix}} & \\ & & & 2 \end{pmatrix}\hbar^2 \begin{pmatrix} |\uparrow\uparrow\rangle \\ |\uparrow\downarrow\rangle \\ |\downarrow\uparrow\rangle \\ |\downarrow\downarrow\rangle \end{pmatrix}$$

$\det|S^2 - \lambda I|$

$\therefore \begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 1 = 0$

$\lambda = 0, 2.$

$\lambda = 0.\ \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$

$\lambda = 2.\ \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$

$$S^2 = \begin{pmatrix} 2 & & & \\ & 0 & & \\ & & 2 & \\ & & & 2 \end{pmatrix} \begin{pmatrix} |\uparrow\uparrow\rangle \\ \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \\ \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \\ |\downarrow\downarrow\rangle \end{pmatrix} \hbar^2$$

↓ 交换顺序

$$S^2 = \begin{pmatrix} 0 & & & \\ & 2 & & \\ & & 2 & \\ & & & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \\ |\uparrow\uparrow\rangle \\ \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \\ |\downarrow\downarrow\rangle \end{pmatrix} \hbar^2$$

$S^2 |s=0, m=0\rangle = s(s+1)\hbar^2 |s, m\rangle$

$\lambda=0$ define $|s=0, m=0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$. singlet

$|s=1, m=1\rangle = |\uparrow\uparrow\rangle$

$\lambda=2$. $|s=1, m=0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$ triplet. 三重简并.

$|s=1, m=-1\rangle = |\downarrow\downarrow\rangle$.

$|s_1, m_1, s_2, m_2\rangle \Rightarrow |s, m, S_{1z}, S_{2z}\rangle$

In General.

$\vec{J} = \vec{J}_1 + \vec{J}_2$

$J_i = J_{1i} + J_{2i}$

$$\left.\begin{aligned} &[J_{1i}, J_{1j}] = i\hbar \varepsilon_{ijk} J_{1k}. \\ &[J_{2i}, J_{2j}] = i\hbar \varepsilon_{ijk} J_{2k}. \\ &[J_{1i}, J_{2j}] = 0 \end{aligned}\right\} \Rightarrow [J_i, J_j] = i\hbar \varepsilon_{ijk} J_k.$$

$J_1^2, J_2^2, J_{1z}, J_{2z} \quad |j_1, j_2, m_1, m_2\rangle$.

m_1 有 $2j_1+1$ 项.
m_2 有 $2j_2+1$ 项.
j_1, j_2 固定时.
共有 $(2j_1+1)(2j_2+1)$ 项.

$J_1^2 |j_1, j_2, m_1, m_2\rangle = j_1(j_1+1)\hbar^2 |j_1, j_2, m_1, m_2\rangle$

$J_2^2 |j_1, j_2, m_1, m_2\rangle = j_2(j_2+1)\hbar^2 |j_1, j_2, m_1, m_2\rangle$

$J_{1z} |j_1, j_2, m_1, m_2\rangle = m_1\hbar |j_1, j_2, m_1, m_2\rangle$

$J_{2z} |j_1, j_2, m_1, m_2\rangle = m_2\hbar |j_1, j_2, m_1, m_2\rangle$

$J^2 . J_1^2 . J_2^2 . J_z .$

$[J^2, J_1^2] = 0 . \quad [J^2, J_2^2] = 0$

$J^2 = J_1^2 + J_2^2 + J_{1+}J_{2-} + J_{1-}J_{2+} + 2J_{1z}J_{2z}$

$J_1^2 |j_1, j_2; j, m\rangle = j_1(j_1+1)\hbar^2 |j_1, j_2; j, m\rangle$

$J_2^2 |j_1, j_2; j, m\rangle = j_2(j_2+1)\hbar^2 |j_1, j_2; j, m\rangle$

$J^2 |j_1, j_2; j, m\rangle = j(j+1)\hbar^2 |j_1, j_2; j, m\rangle$

$J_z |j_1, j_2; j, m\rangle = m\hbar \, |j_1, j_2; j, m\rangle$

$$|j_1, j_2; j, m\rangle = \sum_{m_1, m_2} |j_1, j_2, m_1, m_2\rangle\langle j_1, j_2, m_1, m_2 | j_1, j_2; j, m\rangle$$

$$= \sum_{m_1, m_2} \langle j_1, j_2, m_1, m_2 | j_1, j_2; j, m\rangle \, |j_1, j_2, m_1, m_2\rangle$$

Clebsch-Gordan coefficients.

$(2j_1+1)(2j_2+1).$

$m = -j, \cdots\cdots, j.$

$\langle j_1, j_2, m_1, m_2 | J_z - J_{1z} - J_{2z} | j_1, j_2, j, m\rangle = 0$

$(m - m_1 - m_2)\langle j_1, j_2, m_1, m_2 | j_1, j_2, m_1, m_2\rangle = 0.$

$(2j_1+1)(2j_2+1) = \sum_j (2j+1)$

$|j_1 - j_2| \leq J \leq j_1 + j_2 .$

$\boxed{S_1 = \tfrac{1}{2} . \; S_2 = -\tfrac{1}{2} . \quad \therefore 0 \leq S \leq 1 .}$

$$\therefore (2j_1+1)(2j_2+1) = \sum_{j=j_2-j_1}^{j_1+j_2} (2j+1)$$

assume $j_2 > j_1$.

$$= \frac{1}{2}\left[2(j_2-j_1)+1+2(j_1+j_2)+1\right] \times (2j_1+1)$$

$$= (2j_2+1)(2j_1+1)$$

$|j_1 - j_2| \leq J \leq j_1 + j_2 .$

$\sum_j \sum_m |j_1, j_2, j, m\rangle\langle j_1, j_2, j, m| = \mathbb{I} .$

$\langle j_1, j_2, m_1, m_2 | j_1, j_2, m_1', m_2'\rangle = \delta_{m_1, m_1'} \delta_{m_2, m_2'} .$

$\sum_j \sum_m \langle j_1, j_2, m_1, m_2 | j_1, j_2, j, m\rangle\langle j_1, j_2, j, m | j_1, j_2, m_1', m_2'\rangle = \delta_{m_1, m_1'} \delta_{m_2, m_2'} .$

（从最高项取）

$|j_1, j_2, m_1=j_1, m_2=j_2\rangle = |j_1, j_2, j=j_1+j_2, m=j_1+j_2\rangle.$

$|j_1, j_2, j=j_1+j_2, m=j_1+j_2-1\rangle = \frac{1}{\sqrt{2(j_1+j_2)}} J_- |j_1, j_2, m_1=j_1, m_2=j_2\rangle.$

$J_- = J_{1-} + J_{2-}$

最高项.

$|j_1, j_2, J=j_1+j_2, m=j_1+j_2-1\rangle = \sqrt{\frac{j_1}{j_1+j_2}} |j_1, j_2, m_1=j_1-1, m_2=j_2\rangle + \sqrt{\frac{j_2}{j_1+j_2}} |j_1, j_2, m_1=j_1, m_2=j_2-1\rangle$

$|j_1, j_2, J=j_1+j_2-1, m=j_1+j_2-1\rangle = \sqrt{\frac{j_2}{j_1+j_2}} |j_1, j_2, m_1=j_1-1, m_2=j_2\rangle - \sqrt{\frac{j_1}{j_1+j_2}} |j_1, j_2, m_1=j_1, m_2=j_2-1\rangle$

$J=j_1+j_2-2.$

$J_1=\frac{1}{2}.\ J_2=\frac{1}{2}. \qquad J=1, 0$

$|J=1, m=1\rangle = |\frac{1}{2}, \frac{1}{2}\rangle$

$|J=1, m=0\rangle = \frac{\sqrt{2}}{2}(|\frac{1}{2}, -\frac{1}{2}\rangle + |-\frac{1}{2}, \frac{1}{2}\rangle)$

$|J=1, m=-1\rangle = |-\frac{1}{2}, -\frac{1}{2}\rangle.$

$|J=0, m=0\rangle = \frac{\sqrt{2}}{2}(|\frac{1}{2}, -\frac{1}{2}\rangle - |-\frac{1}{2}, \frac{1}{2}\rangle)$

$j_1=1.\ j_2=1.\quad j=0, 1, 2.$

$|j=2, m=2\rangle = |m_1=1, m_2=1\rangle$

$|j=2, m=1\rangle = \frac{1}{\sqrt{2}}(|m_1=0, m_2=1\rangle + |m_1=1, m_2=0\rangle)$

$|j=2, m=0\rangle = \frac{1}{\sqrt{6}}|m_1=1, m_2=-1\rangle + \sqrt{\frac{2}{3}}|m_1=0, m_2=0\rangle + \frac{1}{\sqrt{6}}|m_1=-1, m_2=1\rangle.$

$|j=2, m=-1\rangle = \frac{1}{\sqrt{2}}(|m_1=0, m_2=-1\rangle + |m_1=-1, m_2=0\rangle)$

$|j=2, m=-2\rangle = |m_1=-1, m_2=-1\rangle.$

$|j=1, m=1\rangle = \frac{1}{\sqrt{2}}(|m_1=0, m_2=1\rangle - |m_1=1, m_2=0\rangle).$

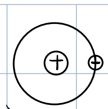
$|j=1, m=0\rangle = \frac{1}{\sqrt{2}}(|m_1=1, m_2=-1\rangle - |m_1=-1, m_2=1\rangle).$

$|j=1, m=-1\rangle = \frac{1}{\sqrt{2}}(|m_1=0, m_2=-1\rangle - |m_1=-1, m_2=0\rangle).$

$|j=0, m=0\rangle = \frac{1}{\sqrt{3}}(|m_1=1, m_2=-1\rangle - |m_1=0, m_2=0\rangle + |m_1=-1, m_2=1\rangle).$

作业：$j_1=1.\ j_2=\frac{3}{2}.$

spin-orbit



$$H = \frac{P^2}{2m} + V(r) - \vec{\mu}\cdot\vec{B}$$

$$\vec{\mu} \propto \vec{S}. \quad \vec{B} \propto \vec{L}$$

Bohr mogaton

$$H_{soc} = -\beta \vec{S}\cdot\vec{L}. \qquad \beta = \frac{\mu_0}{4\pi} g_s \mu_B^2 \times \frac{1}{n^3 a_0^3 l(l+\frac{1}{2})(l+1)}$$

g_s: g factor. 如果自旋. $g=2$.

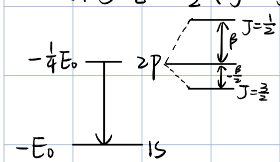
四

For $n=2$, $l=1$, $s=\frac{1}{2}$.

$$\vec{S}\cdot\vec{L} = \left[(\vec{S}+\vec{L})^2 - \vec{S}^2 - \vec{L}^2\right]\frac{1}{2}.$$

$$\vec{J} = \vec{L} + \vec{S}$$

$$\therefore \vec{S}\cdot\vec{L} = \frac{1}{2}(\vec{J}^2 - \vec{S}^2 - \vec{L}^2)$$



$J = |L \pm S| = \frac{1}{2}$ or $\frac{3}{2}$.

$J = \frac{3}{2}$. $H_{soc} = -\frac{\beta}{2}\left(\frac{3}{2}\times\frac{5}{2} - \frac{1}{2}\times\frac{3}{2} - 1\times 2\right) = -\frac{\beta}{2}$

$J = \frac{1}{2}$. $H_{soc} = -\frac{\beta}{2}\left(\frac{1}{2}\times\frac{3}{2} - \frac{1}{2}\times\frac{3}{2} - 1\times 2\right) = \beta$

	S	P	D	F	G
L	0	1	2	3	4

$^{2S+1}L_J$. $\vec{J} = \vec{L} + \vec{S}$

term Symbol

$^{2S+1}P_{\frac{1}{2}}$ $\quad$ $^{2S+1}P_{\frac{3}{2}}$

$S=0$	$S=1$
singlet	triplet